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Note

On the Solution of a Neumann Problem for an Inhomogeneous Laplace Equation*

Frequently, problems encountered in fluid dynamics involve an inhomogeneous Laplace equation, sometimes called a Poisson equation,

$$\Delta\phi = -f, \tag{1}$$

in region Ω. Most commonly, these boundary value problems are completed as Dirichlet problems, that is, the value of the function is specified on the boundary curve (see Fig. 1),

$$\phi = g(s) \tag{2}$$

on σ. This type of problem arises, for example, when determining the stream function ϕ, corresponding to a given vorticity field f. Less commonly, these

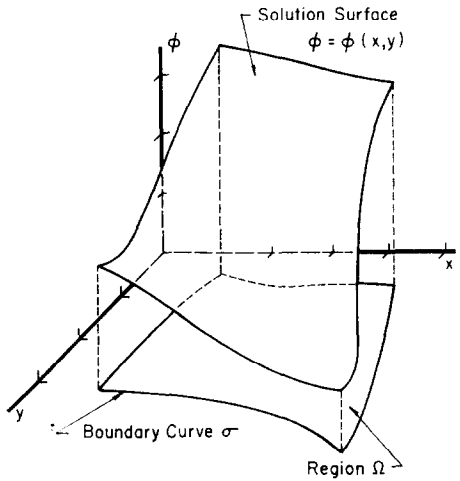

FIG. 1. Solution surface.

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problems are completed as Neumann problems, that is, where the value of the normal derivative is specified on the boundary curve

$$\partial_n \phi = g(s) \tag{3}$$

on σ. This type of problem arises when determining the pressure ϕ, of an incompressible fluid corresponding to a given stream function field. The inhomogeneous function f, is, in this case, a simple function of the stream function as has been demonstrated by Roache [1]. Such a problem also arises when determining the velocity potential ϕ, for a compressible flow in the special case where the effects of compressibility are expressible as a spacial function, f. An application of this type has been shown by one of the authors [2].

It should be noted that for the Neumann problem f and g may not be chosen independently, but rather as a consequence of Green's first identity, they must satisfy the auxiliary condition,

$$\int_{\Omega} f \, dv = \oint_{\sigma} g(s) \, ds \tag{4}$$

if a solution is to exist. Derivation of this condition is found in many texts dealing with partial differential equations, including those of Sneddon [3] and Duff and Naylor [4].

General analytic formulas, giving the solution to each of the aforementioned problems, can be found in mathematical texts such as Duff and Naylor [4]. Such formulas always contain either Green or Neumann functions, which are readily determinable only for the most basic geometric configurations of Ω. Even in the cases where the proper function is available, it is common for the formulas to be given in terms of a slowly convergent, infinite series, which may be either too cumbersome or too lengthy to utilize. Therefore, it is not unusual to approximate the solutions to these problems by finite difference methods. Due primarily to its more frequent occurrence, finite difference methods of approaching the Dirichlet problem have received extensive mention in the literature. In contrast, the number of works directly applicable to the Neumann problem is rather sparse, and these works are widely scattered among the learned journals.

Three finite difference approaches to the Neumann problem are prevalent in the literature. In the approach exemplified by the work of Friedrichs and Keller [5], the difference equations are derived from a variational principle for the difference solution. In the approach exemplified by the work of Bramble and Hubbard [6], special constructions are employed to set up the finite difference boundary conditions, but the interior points are treated in a standard manner. In the approach with which this note is most concerned, standard, centered, second-order accurate, finite difference expressions are utilized both in the interior and on the boundary

of the field. The solution to the resultant linear system is most easily obtained by an iterative relaxation procedure due to the banded nature of the matrix. Such an iterative scheme is not always as straight forward as in the Dirichlet problem.

It is a well-known property of the Neumann problem that, although the shape of its solution surface (Fig. 1) is unique, the location of this solution surface is not fixed in space, as in the Dirichlet problem. There are an infinity of solution surfaces, each differing from the others by an additive constant. This property is the result of the existence of a zero eigenvalue in the solution of the problem. When the finite difference analog of the Neumann problem is formulated, the matrix of the resultant, linear system is singular having rank one less than its order. Mitchell [7] has shown that for the homogeneous equation the solution to the linear system plus the auxiliary condition involves an arbitrary additive constant. This result continues to be valid when the inhomogeneous equation is involved, so that the solution to the finite difference analog is also unique only to within an additive constant. As a result, care must be exercised when formulating an iterative relaxation scheme to approximate the solution to a Neumann problem.

Using the standard iteration scheme,

$$\phi_{i,j}^{N} = \frac{(\Delta x)^2 (\Delta y)^2}{2((\Delta x)^2 + (\Delta y)^2)} \left[\frac{\phi_{i+1,j}^{N-1} + \phi_{i-1,j}^{N-1}}{(\Delta x)^2} + \frac{\phi_{i,j+1}^{N-1} + \phi_{i,j-1}^{N-1}}{(\Delta y)^2} + f \right] \tag{5}$$

to compute a new value at each point on the mesh will yield an approximation to a solution surface. Applying the same formula again will produce another approximation to a solution surface. In this procedure no constraint has been introduced which would require the solution surfaces approximated by successive iterations to be identical. Neither should it be necessary to enter such a constraint when using the Jacobi scheme. It is however convenient for successive iterations to approximate the same solution surface since the change between successive iterations can then be used as an indication of convergence. By noting that two solution surfaces, passing through the same point, must be identical, a method of introducing the desired constraint becomes evident. A point is chosen at which the solution surface will be required to pass through an arbitrary assigned value. New values are then computed at all points on the mesh, including the specially chosen point. The resultant approximation to the solution surface is shifted by adding a constant to the computed value at each point. The magnitude of the constant is chosen to be such as to insure that the value at the special point is equal to the previously assigned value. In this manner, successive iterates may be made to approximate the same solution surface.

At this point, an often proposed alternative method of insuring that successive iterations approximate the same solution surface should be considered. This method assigns a value to a particular point in the mesh and then holds the value

at this point constant through successive iterations. Unlike the method discussed above the value at this point is not recomputed and then brought back to its assigned value by shifting the entire solution surface. Rather, a new value is never computed at this point. It is proposed that since the value at the chosen point is used in the computation of adjacent points, the solution will be tied to one solution surface in much the same manner as in a Dirichlet problem. These authors and others [8] have found this method to be ineffective, largely due to the formation of a singularity at the chosen point. The explanation of this phenomena is deceptively simple. By failing to recompute the value at this point the restriction that the partial differential equation must be satisfied is omitted at this point. Thus the link with any solution surface is lost at this point, and the ineffectiveness of the method is understandable.

Once a Jacobi type iteration scheme, which successively approximates the same solution surface, has been properly developed, a minor modification converts it to the more efficient Gauss–Seidel type of scheme. In this scheme the most recent approximations are used as soon as they become available during an iteration. The approximations to the partial derivatives,

$$\partial_{xx}\phi = (\phi^{N-1}_{i+1,j} + \phi^{N}_{i-1,j} - 2\phi^{N-1}_{i,j})/(\Delta x)^2 \tag{6}$$

$$\partial_{yy}\phi = (\phi^{N-1}_{i,j+1} + \phi^{N}_{i,j-1} - 2\phi^{N-1}_{i,j})/(\Delta y)^2, \tag{7}$$

where N is the current iteration and the field is swept in increasing i and j, now contain values from both the current and the previous iterations. It is clearly necessary that both the current and previous iterations give values approximating the same solution surface, if these expressions are to have the desired significance. By choosing the point which is computed first as the point which will be reduced to a constant value during each iteration, the difference between the two solution surfaces may be computed at the start of the scheme. This difference may then be used to correct the value at each point in the field as it is computed.

$$\phi^{N}_{i,j} = \frac{(\Delta x)^2\,(\Delta y)^2}{2((\Delta x)^2 + (\Delta y)^2)}\left[\frac{\phi^{N-1}_{i+1,j} + \phi^{N}_{i-1,j}}{(\Delta x)^2} + \frac{\phi^{N-1}_{i,j+1} + \phi^{N}_{i,j-1}}{(\Delta y)^2} + f\right] - K \tag{8}$$

where

$$K = \phi^{N}_{\underline{1,1}_J} - \phi^{N-1}_{1,1}$$

and

$$\phi^{N}_{\underline{1,1}_J}$$

is computed from Jacobi scheme.

The technique may be made even more efficient by developing the counterpart of the scheme known as successive over-relaxation (S.O.R.). The extension from the Gauss–Seidel technique to S.O.R. is completely analogous to that given by Smith [9] for a Dirichlet problem.

Fortran computer programs, utilizing both the Jacobi and the Gauss–Seidel type iteration schemes, were developed. Convergence to a solution surface was obtained for a sample problem on a rectangular grid containing 1600 grid points. In comparative runs the Gauss–Seidel type scheme attained the same level of convergence as the Jacobi type scheme in approximately half the number of iterations. No attempt was made to utilize the S.O.R. scheme, due to the considerable uncertainty associated with the computation of the optimum over-relaxation factor.

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